

Formal power series and applications - Tutorial 01

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Exercise 1. Let $\mathbb{K}$ be a field of characteristic 0 and let $f \in \mathbb{K}[[x]]$.

1. Determine a necessary and sufficient condition for f to admit a multiplicative inverse.
2. Does this characterization extend to formal power series over a commutative ring? What about invertibility with respect to the Hadamard product?
3. Determine the multiplicative inverses of $\sum_{n \in \mathbb{N}} x^n$, $(1-x)^3$ and, more generally, $(1-x)^k$ for $k \in \mathbb{N}$.

Solution.

1. $f = \sum_{n \in \mathbb{N}} a_n x^n \in \mathbb{K}[[x]]$ is invertible if and only if $a_0 \neq 0$.
 $\Rightarrow$: If f is invertible, there exist $f^{-1} = \sum_{n \in \mathbb{N}} b_n x^n \in \mathbb{K}[[x]]$ such that $ff^{-1} = 1$. By identifying the coefficients of ff^{-1} and 1, we obtain that $a_0 b_0 = 1$, thus $a_0 \neq 0$.
Alternatively: by contradiction, if $a_0 = 0$ then $f = \sum_{n \geq 1} a_n x^n$ and, for all $g \in \mathbb{K}[[x]]$, $\text{ord}(fg) = \text{ord}(f) + \text{ord}(g) = 1 + \text{ord}(g) > 0 = \text{ord}(1)$. Thus, f cannot be invertible.
 $\Leftarrow$: Conversely, if $a_0 \neq 0$ and since $\mathbb{K}$ is a field, a_0 is invertible. Let us denote its inverse by a_0^{-1} and define $f^{-1} = \sum_{n \in \mathbb{N}} b_n x^n \in \mathbb{K}[[x]]$ inductively as follows. By the identity $ff^{-1} = \sum_{n \in \mathbb{N}} \sum_{i=0}^n a_i b_{n-i} x^n = 1$, we necessarily have $a_0 b_0 = 1$ and thus $b_0 = a_0^{-1}$. Moreover, for all $n \geq 1$, $\sum_{i=0}^n a_i b_{n-i} = 0$ and thus $b_n = a_0^{-1} (-\sum_{i=1}^n a_i b_{n-i})$. Note that each b_n only depend on the definition of the previous b_i for $i < n$.
2. For formal power series over a commutative ring, the characterization becomes: $f \in R[[x]]$ is invertible if and only if $f(0)$ is invertible. The proof remains the same.
For the Hadamard product, the neutral element is the series $\sum_{n \in \mathbb{N}} x^n$ and thus, f is invertible if and only if all its coefficients are invertible.
3. For $\sum_{n \in \mathbb{N}} x^n$, notice that $\sum_{n \in \mathbb{N}} x^n (1-x) = \sum_{n \in \mathbb{N}} x^n - \sum_{n \geq 1} x^n = 1$.
For $(1-x)^3$, by derivating both sides of the equality $\frac{1}{1-x} = \sum_{n \in \mathbb{N}} x^n$, we obtain $\frac{1}{(1-x)^2} = \sum_{n \in \mathbb{N}} (n+1)x^n$. Derivating a second time yields $\frac{2}{(1-x)^3} = \sum_{n \in \mathbb{N}} (n+1)(n+2)x^n$, i.e. $\frac{1}{(1-x)^3} = \sum_{n \in \mathbb{N}} \frac{(n+1)(n+2)}{2} x^n$.

Generalizing the pattern observed previously, we can show by induction that $\frac{1}{(1-x)^k} = \sum_{n \in \mathbb{N}} \binom{n+k-1}{n} x^n$. □

Exercise 2. Determine explicit formulas for the coefficients of $\frac{x^2}{(1-x)(1-2x)}$ and $\frac{1+x}{1-2x+x^2}$.

Solution. For $\frac{x^2}{(1-x)(1-2x)}$, using partial fraction decomposition (i.e. decomposing $\frac{1}{(1-x)(1-2x)}$ as $\frac{a}{1-x} + \frac{b}{1-2x}$ for some $a, b \in \mathbb{Q}$) we obtain:

$$\begin{aligned} \frac{x^2}{(1-x)(1-2x)} &= x^2 \left(\frac{2}{1-2x} - \frac{1}{1-x} \right) \\ &= x^2 \left(2 \sum_{n \in \mathbb{N}} 2^n x^n - \sum_{n \in \mathbb{N}} x^n \right) \\ \frac{x^2}{(1-x)(1-2x)} &= \sum_{n \geq 2} (2^{n-1} - 1) x^n \end{aligned}$$

Alternative combinatorial approach: directly compute the Cauchy product of x^2 , $\frac{1}{1-x}$ and $\frac{1}{1-2x}$.

For $\frac{1+x}{1-2x+x^2} = \frac{1+x}{(1-x)^2}$, we can use Exercise 1 to obtain $\sum_{n \in \mathbb{N}} (2n+1)x^n$. □

Rational series and linear recurrence sequence

The Fibonacci sequence $(F_n)_{n \in \mathbb{N}}$ is defined by $F_0 = 0, F_1 = 1$ and $F_n = F_{n-1} + F_{n-2}$, for all $n \geq 2$.

Exercise 3. Determine the generating function of $(F_n)_{n \in \mathbb{N}}$.

Solution. Let $f = \sum_{n \in \mathbb{N}} F_n x^n$ be the generating function of the Fibonacci sequence.

$$\begin{aligned} f &= \sum_{n \in \mathbb{N}} F_n x^n = F_0 + F_1 x + \sum_{n \geq 2} F_n x^n \\ &= x + \sum_{n \geq 2} F_{n-1} x^n + \sum_{n \geq 2} F_{n-2} x^n \\ &= x + x \sum_{n \geq 2} F_{n-1} x^{n-1} + x^2 \sum_{n \geq 2} F_{n-2} x^{n-2} \\ &= x + x(f - F_0) + x^2 f \\ f &= \frac{x}{1-x-x^2} \end{aligned} \quad \square$$

The Fibonacci sequence is a particular case of *linear recurrence sequences*, i.e. sequences $(a_n)_{n \in \mathbb{N}}$ satisfying a recurrence of the form $a_n = \sum_{i=1}^k c_i a_{n-i}$ for some constants $c_1, \dots, c_k$.

Exercise 4. Show that $(a_n)_{n \in \mathbb{N}}$ is a linear recurrence sequence if and only if its generating function is rational.

Solution. Let $f = \sum_{n \in \mathbb{N}} a_n x^n$ be the generating function of $(a_n)_{n \in \mathbb{N}}$.

If $(a_n)_{n \in \mathbb{N}}$ is a linear recurrence sequence then, by definition, there exist $N \geq k$ such that, for all $n > N$, $a_n = \sum_{i=1}^k c_i a_{n-i}$.

For all $i \leq N$, let $P_i = \sum_{n=0}^i a_n x^n \in \mathbb{K}[x]$. Then,

$$\begin{aligned} f &= P_N + \sum_{n>N} a_n x^n = P_N + \sum_{n>N} \left(\sum_{i=1}^k c_i a_{n-i} \right) x^n \\ &= P_N + \sum_{i=1}^k c_i \sum_{n>N} a_{n-i} x^n \\ &= P_N + \sum_{i=1}^k c_i x^i \sum_{n>N} a_{n-i} x^{n-i} \\ &= P_N + \sum_{i=1}^k c_i x^i (f - P_{N-i}) \\ f &= \frac{P}{Q} \quad \left(\text{where } P = P_N - \sum_{i=1}^k c_i x^i P_{N-i} \text{ and } Q = 1 - \sum_{i=1}^k c_i x^i \right) \end{aligned}$$

Thus, f is a rational series.

Conversely, any rational series can be written in the form of the last equality, and running the calculations in reverse yields the linear recurrence relation verified by $(a_n)_{n \in \mathbb{N}}$. $\square$

Exercise 5. The exponential generating function of a sequence $(a_n)_{n \in \mathbb{N}}$ is defined as the formal power series $\sum_{n \in \mathbb{N}} a_n \frac{x^n}{n!}$.

1. Show that $(a_n)_{n \in \mathbb{N}}$ is a linear recurrence sequence if and only if its exponential generating function satisfies a linear differential equation with constant coefficients.
2. Deduce a closed form for the terms of the Fibonacci sequence from the linear differential equation verified by its exponential generating function.

Solution.

1. The idea is the same as in Exercise 4, using derivations to shift the coefficients of f instead.

Let $f = \sum_{n \in \mathbb{N}} a_n \frac{x^n}{n!}$ be the exponential generating function of $(a_n)_{n \in \mathbb{N}}$.

If $(a_n)_{n \in \mathbb{N}}$ is a linear recurrence sequence, there exist $k \in \mathbb{N}$ such that, for all $n \in \mathbb{N}$, $a_{n+k} = \sum_{i=1}^k c_i a_{n+k-i}$.

Note that, for all $i \in \mathbb{N}$, $f^{(i)} = \sum_{n \in \mathbb{N}} a_{n+i} \frac{x^n}{n!}$. Then,

$$\begin{aligned} f^{(k)} &= \sum_{n \in \mathbb{N}} a_{n+k} \frac{x^n}{n!} \\ &= \sum_{n \in \mathbb{N}} \left(\sum_{i=1}^k c_i a_{n+k-i} \right) \frac{x^n}{n!} \\ &= \sum_{i=1}^k c_i \sum_{n \in \mathbb{N}} a_{n+k-i} \frac{x^n}{n!} \\ f^{(k)} &= \sum_{i=1}^k c_i f^{(k-i)} \end{aligned}$$

Running the calculations in reverse yields the converse.

- Using the previous question, we know that the exponential generating function $f = \sum_{n \in \mathbb{N}} F_n \frac{x^n}{n!}$ of the Fibonacci sequence verifies the linear differential equation $f'' - f' - f = 0$.

The characteristic polynomial of the differential equation is $x^2 - x - 1$ whose roots are $\varphi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$. Its solutions are then of the form $f = ae^{\varphi x} + be^{\psi x}$ for some $a, b \in \mathbb{Q}$. Using the initial conditions $f(0) = F_0 = 0$ and $f'(0) = F_1 = 1$, we obtain the system of linear equations:

$$\begin{cases} a + b = 0 \\ \varphi a + \psi b = 1 \end{cases}$$

Solving the system, we find that $f = \frac{1}{\sqrt{5}}(e^{\varphi x} - e^{\psi x}) = \sum_{n \in \mathbb{N}} \frac{\varphi^n - \psi^n}{\sqrt{5}} x^n$, and thus $F_n = \frac{\varphi^n - \psi^n}{\sqrt{5}}$. □

Deciding whether a given linear recurrence sequence contains a zero term is a long standing open problem known as the *Skolem problem*.

In the multivariate setting, the generating function of a linear recurrence sequence is not necessarily rational [1]. Moreover, we can show that the following “analogue” of the Skolem problem is undecidable.

Exercise 6. Let $k \geq 9$. Show that the following decision problem is undecidable: given a rational series $f \in \mathbb{Q}[[x_1, \dots, x_k]]$, does there exist $n_1, \dots, n_k \in \mathbb{N}$ such that $[x_1^{n_1}, \dots, x_k^{n_k}]f = 0$?

Hint: Hilbert’s tenth problem (given a polynomial $p \in \mathbb{Q}[x_1, \dots, x_k]$, does there exist $n_1, \dots, n_k \in \mathbb{N}$ such that $p(n_1, \dots, n_k) = 0$) is undecidable for 9 or more variables.

Solution. This result is due to Litow [2].

To get a reduction from Hilbert’s tenth problem, we need to define a computable function mapping each polynomial $p \in \mathbb{Q}[x_1, \dots, x_k]$ to a rational series $f_p \in \mathbb{Q}[[x_1, \dots, x_k]]$ such that $[x_1^{n_1}, \dots, x_k^{n_k}]f_p = p(n_1, \dots, n_k)$ for all $n_1, \dots, n_k \in \mathbb{N}$.

We define this function on all the monomials $p = x_1^{r_1} \dots x_k^{r_k}$ for all $r_1, \dots, r_k \in \mathbb{N}$ and the result follows directly from its linear extension since both rationality and the desired property are preserved by linear combinations.

Observe that, in the univariate case, applying a derivation followed by a multiplication by x to $\frac{1}{1-x}$, we obtain the series $\sum_{n \in \mathbb{N}} nx^n$. Iterating this operation, which we denote by d_x , we can then define, for all monomial $p = x^r \in \mathbb{Q}[x]$ a rational series $f_p = d_x^r \left(\frac{1}{1-x} \right) \in \mathbb{Q}[[x]]$ such that $[x^n]f_p = n^r = p(n)$ for all $n \in \mathbb{N}$.

Generalizing this idea to the multivariate case, define the operations ∂_i by $\partial_i(f) = x_i \frac{\partial f}{\partial x_i}$ for all $1 \leq i \leq k$ and $f \in \mathbb{Q}[[x_1, \dots, x_k]]$. We can then define, given any monomial $p = x_1^{r_1} \dots x_k^{r_k}$, the rational series $f_p = \partial_1^{r_1} \dots \partial_k^{r_k} \left(\frac{1}{(1-x_1) \dots (1-x_k)} \right)$ which verifies the desired property. $\square$

References

- [1] Mireille Bousquet-Mélou and Marko Petkovsek. Linear recurrences with constant coefficients: the multivariate case. *Discret. Math.*, 225(1-3):51–75, 2000.
- [2] Bruce E. Litow. A note on commutative multivariate rational series. *Inf. Process. Lett.*, 87(6):283–285, 2003.