Languages, automata and computation II
Homework 1

Problems: deadline ~~17/11/2023~~ 24/11/2023

Problem 1 (Learning regular separators)**.** Consider the following problem. Teacher knows two disjoint regular languages $L, M \subseteq \Sigma^*$ and Learner wants to find a regular separator, i.e., a language $S \subseteq \Sigma^*$ including L and disjoint from M. There are two kind of queries. 1) Learner gives a word $w \in \Sigma^*$ to Teacher, who answers "in L", "in M" or "not in $L \cup M$". 2) Learner gives a (DFA recognising a) separator candidate S to Teacher, who answers either "yes" if it separates L, M, or in case it doesn't, Teacher answers "no" and provides either a counter-example to $L \subseteq S$ or to $M \cap S = \emptyset$. We assume that counter-examples returned by Teacher are of minimal size.

Give a learning protocol for this problem, with polynomially many queries in the sizes of minimal DFAs for L, M. Observe that in the special case when L is the complement of M, this problem corresponds to the original learning setup for the Angluin algorithm.

Problem 2 (Finite-valued rational functions, part 1)**.** Let $L_1, \ldots, L_k$ a partition of Σ^* into regular languages, and consider weights $q_1, \ldots, q_k \in \mathbb{Q}$. Show that the following function $f : \Sigma^* \to \mathbb{Q}$ is rational:

$$\text{for every } w \in \Sigma^*: \quad f(w) = \begin{cases} q_1 & \text{if } w \in L_1, \\ \vdots & \\ q_k & \text{if } w \in L_k. \end{cases}$$

Problem 3. Recall that a word $w = a_0 \cdots a_n \in \{0,1\}^*$ encodes a number $[w]_2$ under the least binary digit first encoding

$$[w]_2 = a_0 + a_1 \cdot 2 + \cdots + a_n \cdot 2^n.$$

Let $f : \mathbb{N} \to \mathbb{Q}$ be a rational function. Show that the following function $g : \{0,1\}^* \to \mathbb{Q}$ is recognisable by a polynomial automaton:

$$g(w) = f([w]_2), \quad \text{for every } w \in \{0,1\}^*.$$

Star problems: deadline 22/12/2023

(*) Problem 4 (Finite-valued rational functions, part 2)**.** (This is the converse of Problem 2.) Let f be a rational function taking only finitely many values. Show that for each value $q \in \mathbb{Q}$, the inverse image $f^{-1}(q)$ (the set of words which f maps to q) is a regular language. *Hint: Consider the q-finite representation of f.*

Open problems: end of the semester

Open problem 1. Prove or disprove: For every function $f : \Sigma^* \to \mathbb{Q}$ computed by a polynomial automaton, there is a polynomial automaton that computes the reverse function defined by

$$a_1 \cdots a_n \quad \mapsto \quad f(a_n \cdots a_1).$$

Disclaimer: we do not know the answer, although we believe that the statement is most likely false, i.e. there is a counterexample.

Open problem 2. Is the following problem is decidable? We are given a polynomial automaton computing a function $f : \Sigma^* \to \mathbb{Q}$, and we want to know if the following function gives only zero outputs:

$$a_1 \cdots a_n \quad \mapsto \quad f(a_1 \cdots a_n a_n \cdots a_1).$$

Disclaimer: we do not know the answer, although we believe that the problem is decidable.