## Languages, automata and computation II Tutorial 8 - MSO and automata on infinite trees

## Winter semester 2023/2024

Exercise 1. Determine whether the following languages of finite trees are regular:

- 1. Trees with an even number of nodes.
- 2. Trees with an even number of nodes labelled by a.
- 3. Trees where every subtree contains an a.
- 4. Trees encoding Boolean expressions evaluating to true. The alphabet contains two symbols  $\top$ ,  $\bot$  of arity 0, one symbol  $\neg$  of arity 1, and two symbols  $\land$ ,  $\lor$  of arity 2.
- 5. Trees where every leaf is at the same depth (balanced trees).

Solution:  $\Box$ 

**Exercise 2.** 1. Show that nondeterministic finite tree automata have the same expressive power as deterministic bottom-up finite tree automata.

- 2. Show that deterministic bottom-up finite tree automata are more expressive than deterministic top-down finite tree automata.
- 3. Show that deterministic bottom-up parity tree automata are less expressive than nondeterministic parity tree automata.

Solution: 1. The subset construction works straight away.

- 2. Consider the language  $L = \{a(b, c), a(c, b)\}$ . Any deterministic top-down finite tree automaton recognising L accepts also the tree a(b, b), which is a contradiction.
- 3. The language of infinite trees containing finitely many occurrences of letter a is recognisable by a nondeterministic parity tree automaton, but not by a bottom-up deterministic one.

**Exercise 3.** Show that nonemptiness of nondeterministic parity tree automata is decidable. What is the computational complexity of this problem?

Solution: Given a NPTA A one can build in PTIME a finite game graph with parity winning condition. Viceversa, for every such game one can build in PTIME a NPTA A which has nonempty language iff Player 0 wins. Thus NPTA nonemptiness and parity games on finite graphs are PTIME-equivalent.

**Exercise 4.** Consider finite word structures  $(\{1, \ldots, n\}, \leq, P_a, P_b)$  where  $P_a, P_b$  are two unary relation symbols. Show a class of finite word structures definable in existential second-order logic which is not MSO definable (thus not regular). Can this be done without unary relation symbols  $P_a, P_b$ ?

Solution: The finite word structures corresponding to the nonregular language  $a^nb^n$  are definable in existential second-order logic. It suffices to existentially quantify over a binary relation  $R \subseteq \{1, \ldots, 2n\}^2$  and axiomatise that it is a well-nested matching between the two half of the domain, where the first half has label a and the second has label b.

In fact we do not need  $P_a, P_b$  to define nonregular languages in existential second-order logic: The set of finite word structures of composite size (nonprime) is not regular, but definable in this logic. One guesses two subsets of the domain  $X, Y \subseteq \{1, \ldots, n\}$ , each of size at least 2, and a bijection from  $\{1, \ldots, n\}$  to  $X \times Y$ .

**Exercise 5.** A tree is *regular* if it contains only finitely many distinct subtrees. Show that a nonempty regular language of infinite trees contains a regular tree.

Solution: Consider the nonemptiness game played on the infinite binary tree. Since Player 0 wins, she has a memoryless winning strategy. Applying this strategy constructs a regular tree accepted by the automaton.

**Exercise 6.** Fix a regular language of  $\omega$ -words  $L \subseteq \{a, b\}^{\omega}$ . Show that the language of infinite trees where every branch belongs to L is regular.

Solution: From a DPA recognising L construct a top-down deterministic parity tree automaton that checks that every branch belongs to L.

**Exercise 7.** Construct a language of infinite trees recognised by a NPTA but not by a nondeterministic Büchi tree automaton, and prove that this is the case.

Solution: Consider the language of infinite trees s.t. every branch contains only finitely many a's. This language is even deterministic coBüchi recognisable. A suitable pumping argument shows that the Büchi acceptance condition cannot recognise this language.

- **Exercise 8.** 1. Show that the structure  $(\mathbb{N}^*, \preceq)$  has a decidable MSO theory, where  $\preceq$  is the prefix order. *Hint: Reduce to the MSO theory of the infinite binary tree.* 
  - 2. Show that the structure  $(\mathbb{N}^*, \leq_{lex})$  has a decidable MSO theory, where  $\leq$  is the lexicographic order.
  - 3. Conclude that  $(\mathbb{Q}, \leq)$  has a decidable MSO theory.

Solution: We encode  $w \in \mathbb{N}^*$  as a node of the binary tree  $[w] \in \{L, R\}^*$ . For instance  $[\varepsilon] = \varepsilon$  and [21] = LLRL. In general,  $[n_1 \cdots n_k] = L^{n_1}RL^{n_2}R\cdots L^{n_k}$ . The image of the encoding is an MSO-definable subset of the binary tree. The prefix order  $\leq$  on  $\mathbb{N}^*$  is the R-descendant relation on the binary tree, which is MSO-definable. Consequently, every MSO formula of the original structure can be encoded as an MSO formula of the binary tree.

The lexicographic order is encoded as the descendant relation.  $\,$ 

The rationals with order are isomorphic  $((0,1) \cap \mathbb{Q}, \leq)$ , which can be MSO-encoded in  $(\{0,1\}^*, \leq_{\text{lex}})$ .

**Exercise 9.** Show that Gale-Steward games with an  $\omega$ -regular winning condition are decidable.

Solution: Reduce to MSO on infinite trees or nonemptiness of NPTA.  $\Box$