

Languages, automata and computation II

Tutorial 2 – WQO

Winter semester 2024/2025

A binary relation \preceq on a set X is **quasi-order** if it is reflexive and transitive. An antichain is a subset of X whose elements are pairwise incomparable w.r.t. \preceq .

Exercise 1. Let (X, \preceq) be a quasi-order. Prove that the following conditions are equivalent.

- (a) Any infinite sequence a_1, a_2, \dots of elements of X contains a *domination* $a_i \preceq a_j$ for some indexes $i < j$.
- (b) X does not contain any infinite antichain and is *well-founded* (i.e., there are no infinite strictly decreasing sequences).
- (c) Any infinite sequence a_1, a_2, \dots of elements of X contains an infinite nondecreasing subsequence, i.e., there are indices $i_1 < i_2 < \dots$ with $a_{i_1} \preceq a_{i_2} \preceq \dots$.
- (d) Every upward closed set is the upward closure of a finite set. (The *upward closure* of a set X is the set of elements that dominate some element of X ; X is *upward closed* if it is equal to its upward closure.)
- (e) Every nondecreasing chain of upward closed sets $U_1 \subseteq U_2 \subseteq \dots \subseteq X$ is finite.

Whenever any of the conditions above holds we call the relation \preceq a **well quasi-order** or **WQO**.

Exercise 2. Decide which of the following quasi orders are WQO.

- (a) (\mathbb{N}, \leq) ;
- (b) (\mathbb{Z}, \leq) ;
- (c) $(\mathbb{N}, |)$, where $|$ is the divisibility relation;
- (d) (X, \preceq) where X is finite and \preceq is any quasi-order on X ;
- (e) $(\mathbb{N}^2, \leq_{\text{lex}})$, where \leq_{lex} is the lexicographic order;
- (f) $(\{a, b\}^*, \leq_{\text{lex}})$, where \leq_{lex} is the lexicographic order on words;
- (g) $\{a, b\}^*$ with the prefix order;
- (h) $\{a, b\}^*$ with the infix order;

(i) intervals of \mathbb{N} with the following order:

$$[a, b] \preceq [c, d] \quad \text{if } b < c \vee (a = c \wedge b \leq d);$$

(j) graphs with the subgraph order (remove some edges and some vertices).

Exercise 3 (Dickson's lemma). Let (X, \leq_X) , (Y, \leq_Y) be WQOs and consider the *product order* $(X \times Y, \leq_{X \times Y})$ where

$$(x, y) \leq_{X \times Y} (x', y') \quad \text{iff} \quad (x \leq_X x' \wedge y \leq_Y y').$$

Show that the product order is a WQO. Conclude that (\mathbb{N}^d, \leq) is a WQO, where \leq is the pointwise partial order.

Exercise 4. Let $\mathcal{P}_{\text{fin}}(X)$ be the set of all finite subsets of X . For a quasi-order \preceq on X , define a quasi-order $\tilde{\preceq}$ on $\mathcal{P}_{\text{fin}}(X)$ as follows: $A \tilde{\preceq} B$ if and only if there is an injection $f : A \rightarrow B$ such that $a \preceq f(a)$ for all $a \in A$. Prove that if (X, \preceq) is a WQO, then also $(\mathcal{P}_{\text{fin}}(X), \tilde{\preceq})$ is a WQO.

Exercise 5 (Higman's lemma). Let (X, \preceq) be a WQO and consider the *subword order* (a.k.a. *scattered subsequence order*) on words $\preceq^* \subseteq X^* \times X^*$. Prove that (X^*, \preceq^*) is a WQO.