Languages, automata and computation II
Tutorial 3 – Applications of well-quasi orders

Winter semester 2024/2025

Regular languages

Exercise 1. Consider the set of finite words well-quasi ordered by the subword relation $(\Sigma^*, \sqsubseteq)$. Show that *every* downward closed language over Σ is regular.

Lossy rewrite systems

Exercise 2. A *rewrite system* over a finite alphabet Σ is a finite set of pairs $u \to v$ with $u, v \in \Sigma^*$. Consider the least reflexive and transitive congruence $\to^*$ on Σ^* containing $\to$. A rewrite system is *lossy* if it contains transitions $a \to \varepsilon$ for every $a \in \Sigma$. Show that the relation $\to^*$ is decidable when $\to$ is lossy.

Vector addition systems

Exercise 3. Let $\mathcal{V}$ be a d-dimensional VASS and consider a target configuration $t \in P \times \mathbb{N}^d$, where P is the set of states. Show that one can compute the set of *all configurations* s which can cover t.

Exercise 4. Let $\mathcal{V}$ be a d-dimensional VAS and consider a source configuration $s \in \mathbb{N}^d$. Show that one decide whether there are only *finitely many* configurations reachable from s.

Exercise 5. Let $\mathcal{V}$ be a d-dimensional VAS and consider a source configuration $s \in \mathbb{N}^d$. Show that for any coordinate $k \in \{1, \ldots, d\}$ it is decidable whether there exists a number $n \in \mathbb{N}$ such that every configuration t reachable from s has the kth coordinate bounded by n.

Exercise 6. Show that a VASS of dimension d can be simulated by a VAS (without states) of dimension $d + 3$.

Vector addition systems over $\mathbb{Z}$

This part is about VASSes but unrelated with well quasi orders. We show that reachability in VASSes is considerably simpler if we relax the requirement that counters cannot become negative.

Exercise 7. Let a $\mathbb{Z}$-*VASS* of dimension $d \in \mathbb{N}$ be a pair (Q, T) where Q is a finite set of states and $T \subseteq Q \times \mathbb{Z}^d \times Q$ be a finite set of transitions. The semantics is as in VASS, except that now configurations are in $Q \times \mathbb{Z}^d$ (instead of the more restrictive $Q \times \mathbb{N}^d$). Show that reachability is decidable for $\mathbb{Z}$-VASSes.

Strassen's matrix multiplication algorithm

This section is unrelated with well-quasi orders. We begin with a simpler problem.

Problem 1. Consider three complex numbers $a = a_1 + a_2 i, b = b_1 + b_2 i, c = c_1 + c_2 i \in \mathbb{C}$. The naive multiplication algorithm would compute the product $c = a \cdot b$ as

$$\begin{aligned} c_1 &= a_1 \cdot b_1 - a_2 \cdot b_2, \\ c_2 &= a_1 \cdot b_2 + a_2 \cdot b_1, \end{aligned}$$

which uses four multiplications in $\mathbb{R}$. Find a more efficient algorithm that uses only three multiplications and any number of additions.

Consider 2×2 matrices of rational numbers

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, B = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}, C = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \in \mathbb{Q}^{2\times 2}.$$

The naive multiplication algorithm would compute the product $C = A \cdot B$ as

$$\begin{aligned} c_{11} &= a_{11} \cdot b_{11} + a_{12} \cdot b_{21}, \\ c_{12} &= a_{11} \cdot b_{12} + a_{12} \cdot b_{22}, \\ c_{21} &= a_{21} \cdot b_{11} + a_{22} \cdot b_{21}, \\ c_{22} &= a_{21} \cdot b_{12} + a_{22} \cdot b_{22}, \end{aligned}$$

which uses 8 multiplications (we do not care about additions). When applied recursively, this yields the following formula for the number of ring multiplications used in order to compute the product of two $n \times n$ matrices A, B

$$M(n) \leq O(n^2) + 8 \cdot M(n/2).$$

From this we obtain a complexity upper bound $M(n) \leq O(n^3)$ for naive matrix multiplication. Strassen's algorithm uses only 7 multiplications by computing the following products:

$$\begin{aligned} m_1 &= (a_{11} + a_{22}) \cdot (b_{11} + b_{22}), \\ m_2 &= (a_{21} + a_{22}) \cdot b_{11}, \\ m_3 &= a_{11} \cdot (b_{12} - b_{22}), \\ m_4 &= a_{22} \cdot (b_{21} - b_{11}), \\ m_5 &= (a_{11} + a_{12}) \cdot b_{22}, \\ m_6 &= (a_{21} - a_{11}) \cdot (b_{11} + b_{12}), \\ m_7 &= (a_{12} - a_{22}) \cdot (b_{21} + b_{22}). \end{aligned}$$

The number of ring multiplications for the improved algorithm satisfies

$$M(n) \leq O(n^2) + 7 \cdot M(n/2).$$

and thus $M(n) \leq O(n^{\log_2 7})$, where $\log_2 7 \approx 2.81$.