LINEAR-RECURSIVE SEQUENCES & INCLUSION of UNAMBIGUOUS REGISTER AUTOMATA

Lorenzo Clemente (U. Warsaw)

joint work with

Corentin Barloy (U. Lille)

PLAN

- Unambiguous register automata over equality data $(\mathbb{A}, =)$.
- Universality, equivalence, inclusion problems.
  - inclusion $\xrightarrow{\text{REDUCES TO}}$ universality (!).
  - universality $\xrightarrow{\text{REDUCES TO}}$ zeroness of linrec.
- Bidimensional linrec sequences $\mathbb{N} \times \mathbb{N} \longrightarrow \mathbb{N}$.
  - Modelled with skew polynomials.
  - Decidability of the zeroness problem:
    - With elementary complexity via elimination.
    - In EXPTIME via Hermite normal form.

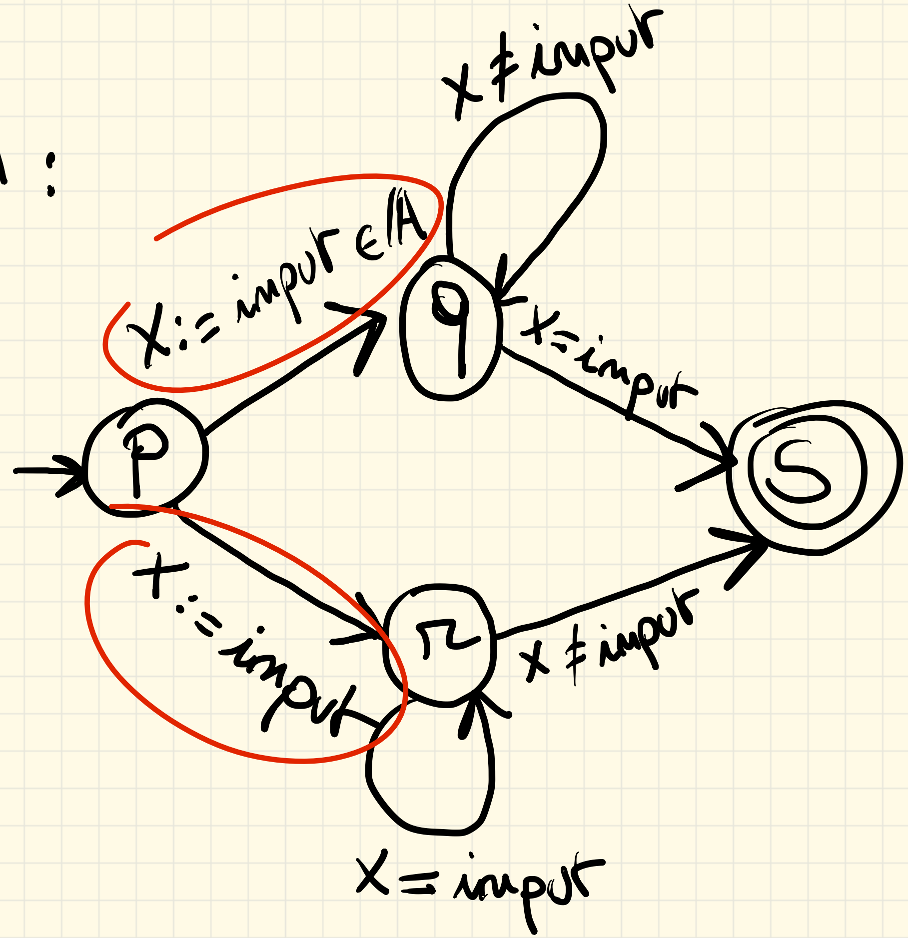
REGISTER AUTOMATA over $(\mathbb{A}, =)$

[Kaminski & Francez TCS'94]

$$L = \{ a b^* a , \; a a^* b \mid a, b \in \mathbb{A}, \; a \neq b \}$$

infinite input alphabet

A :



- one register $x \in \mathbb{A} \cup \{\perp\} =: \mathbb{A}_\perp$
- nondeterministic (in p).
- $L(A) \neq L$.
- Non-guessing : x is either $\perp$ or it stores a value from the input.

ALGORITHMIC ANALYSIS of REGISTER AUTOMATA

Emptiness problem $L(A) = \emptyset$?

→ decidable [1].

→ PSPACE-complete [2].

Universality problem $L(A) = \mathbb{A}^*$?

→ undecidable for ≥ 2 registers [3].

→ decidable for $1\frac{1}{2}$ registers [1]

and non-primitive recursive [2].

Equivalence $L(A) = L(B)$ & Inclusion $L(A) \subseteq L(B)$ similar.

[1] Kaminski & Francez TCS'94

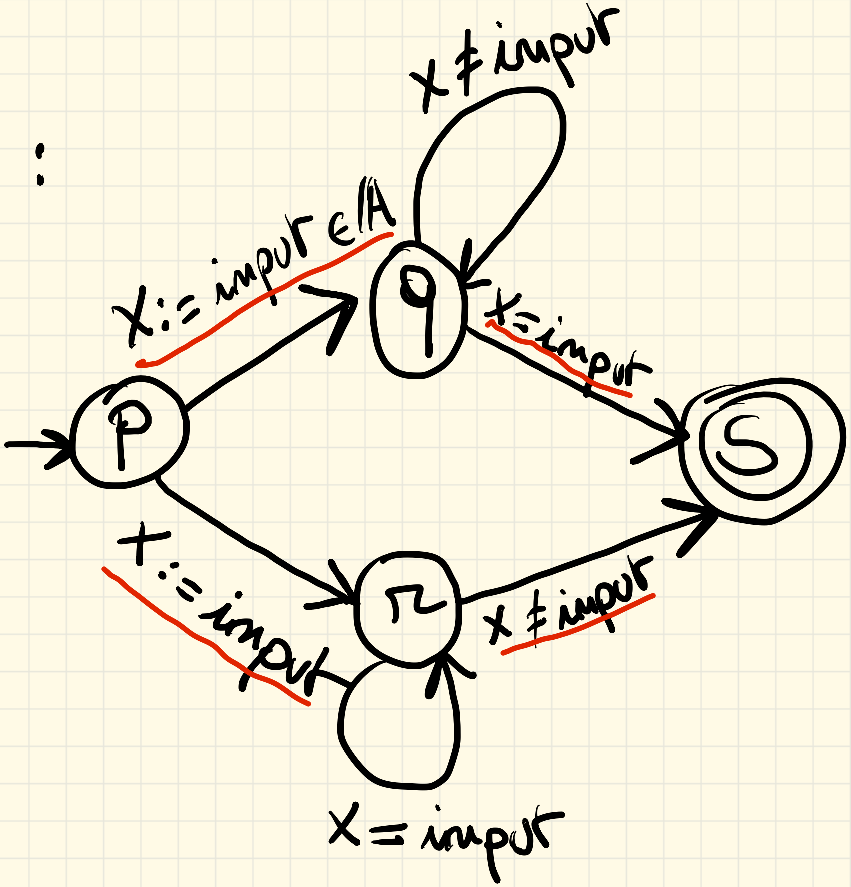
[2] Demri & Lazić ToCL'09

[3] Neven, Schwentick & Vianu ToCL'04

UNAMBIGUOUS REGISTER AUTOMATA

$\boxed{\text{each input induces} \leq 1 \text{ accepting run.}}$

A :



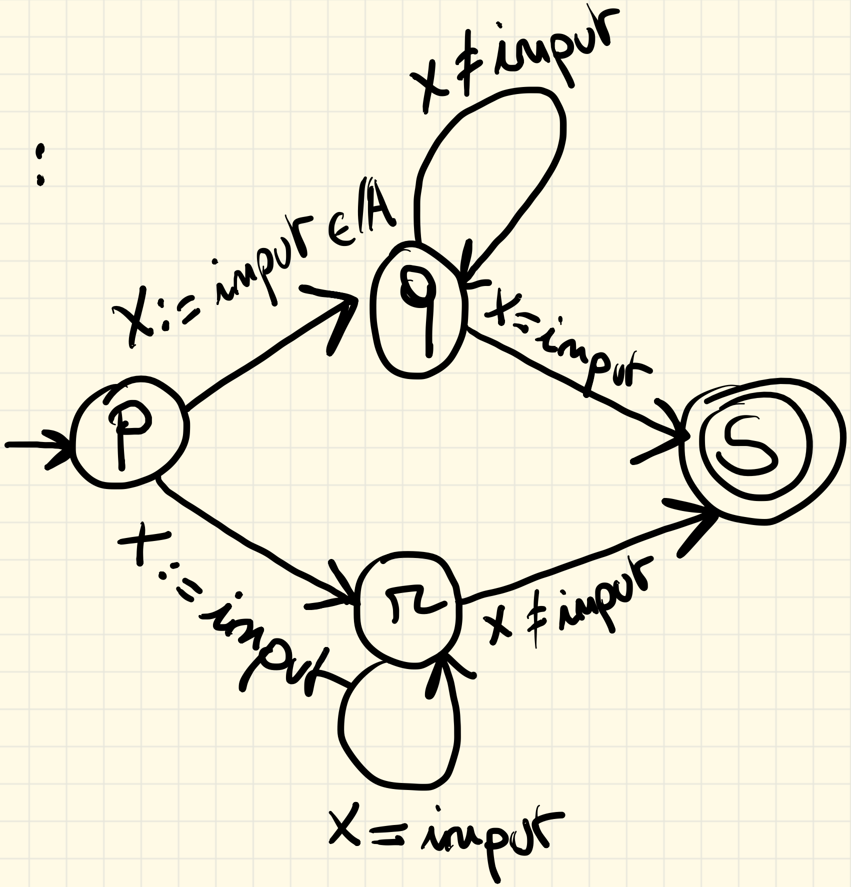
UNAMBIGUOUS

$\cancel{L(A) = \{ a b^* a , a a^* b \mid a , b \in \mathbb{A} , a \neq b \}}$

UNAMBIGUOUS REGISTER AUTOMATA

$\boxed{\text{each input induces } \leq 1 \text{ accepting run.}}$

A :



UNAMBIGUOUS

$L(A) = \{ a b^* a , a a^* b \mid a, b \in \mathbb{A} , a \neq b \}$

L = " Some $a \in \mathbb{A}$ appears twice ".

INHERENTLY AMBIGUOUS

↳ any B s.t. L(B) = L must be ambiguous.

UNIVERSALITY of UNAMBIGUOUS REGISTER AUTOMATA

- Decidable in 2-EXPSPACE [1].
  → via the construction of a pruned reachability graph.

- Long tradition of decidability for unambiguous models:
  → universality of unambiguous finite automata in PTIME [2,3].
  → universality of unambiguous context-free grammars is decidable [4], even in PSPACE [5].

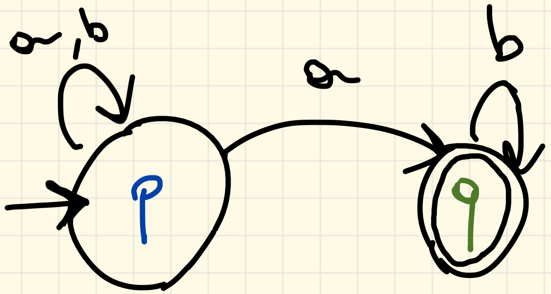
Via the "counting approach".

[1] Mottet & Quaas STACS'19.
[2] Schützenberger IC'61.
[3] Stearns & Hunt SFCS'81.
[4] Salomaa & Soittola '78.
[5] C. VPT'20.

THE COUNTING APPROACH

Unambiguity $\Rightarrow$ bijection between $L(A)$ and $\underbrace{\text{initial \& accepting runs}}_{\text{COUNT RUNS!}}$

FINITE AUTOMATA, alphabet $\Sigma = \{a, b\}$



$f_p(n) :=$ # initial runs of length n to p.

$f_q(n) :=$ " " " to q.

$$\underbrace{\begin{aligned} &f_p(0) = 1, \quad f_q(0) = 0 \\ &f_p(n+1) = 2 \cdot f_p(n) \\ &f_q(n+1) = f_p(n) + f_q(n) \end{aligned}}_{\text{C-recursive}}$$

linear with constant coefficients

$L(A)$ universal

$\Leftrightarrow$ $\longleftarrow$ unambiguity

$f_q(n) = 2^n$

$\Leftrightarrow$

$$\underbrace{f_q(n) - 2^n = 0}_{\text{ZERONESS PROBLEM}}$$

THE COUNTING APPROACH - DATA LANGUAGES

What to count?

TAKE 1: # runs / words $w \in L \subseteq \mathbb{A}^*$ of length n? ∞.

$\mathbb{A}^n$ is infinite but ORBIT-FINITE:

$$\mathrm{ORBIT}(a_1 \cdots a_n) = \{\alpha(a_1) \cdots \alpha(a_n) \mid \text{for some bijection } \alpha : \mathbb{A} \to \mathbb{A}\}$$

Example: $$\mathbb{A}^2 = \underbrace{\{aa \mid a \in \mathbb{A}\}}_{\mathrm{ORBIT}(aa)} \cup \underbrace{\{ab \mid a, b \in \mathbb{A}, a \neq b\}}_{\mathrm{ORBIT}(ab)}.$$

$\boxed{\text{TAKE 2: \# orbits of runs / words of length } n\text{?}}$

$|\overset{\text{orbits}}{(\mathbb{A}^n)}| = n\text{-th Bell number } B(n).$

no "simple" recursive equation for $B(n)$.

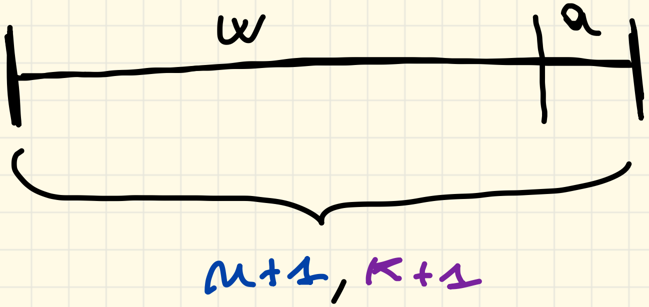
THE COUNTING APPROACH – DATA LANGUAGES

TAKE 3: # orbits of runs/words of len n & k distinct data values

EXAMPLE: $f_{\mathbb{A}^*}(n,k)$ = "# orbits of all words of len n, width k".

$$f_{\mathbb{A}^*}(0,0) = 1 \;; \text{ for } n,k \geq 0 : \; f_{\mathbb{A}^*}(n+1,0) = f_{\mathbb{A}^*}(0,k+1) = 0,$$

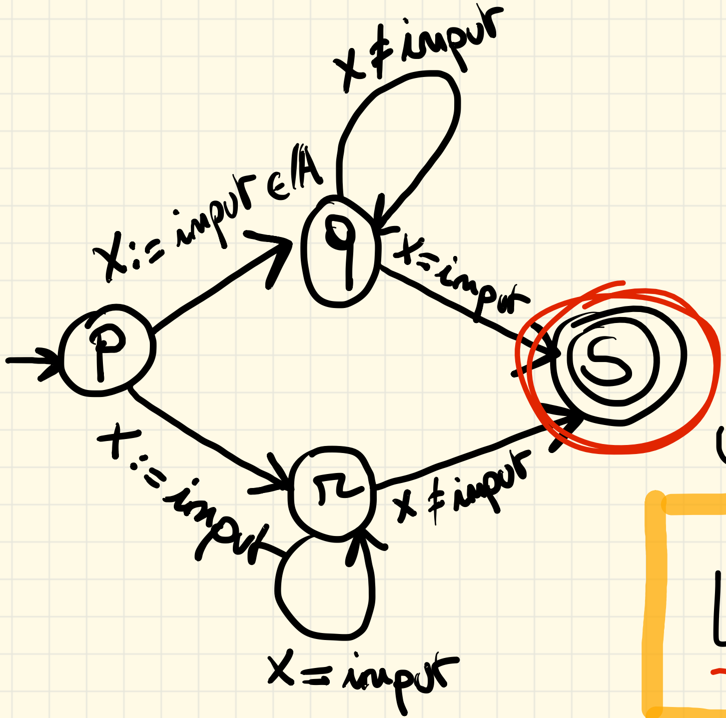
$$f_{\mathbb{A}^*}(n+1,k+1) = \underbrace{f_{\mathbb{A}^*}(n,k)}_{\substack{\text{CASE 1} \\ a \notin w \\ \text{fresh}}} + \underbrace{(k+1)\cdot f_{\mathbb{A}^*}(n,k+1)}_{\substack{\text{CASE 2} \\ a \in w \\ \text{not fresh}}}$$



$f_{\mathbb{A}^*}(n,k)$ = Stirling numbers of the 2nd kind $S(n,k)$.

THE COUNTING APPROACH – DATA LANGUAGES

$f_x(n,k)$ = #orbits of initial runs of len n & k distinct data values ending in x



$$f_p(0,0)=1,\quad f_q(0,0)=f_r(0,0)=f_s(0,0)=0$$

$$f_x(0,k+1)=f_x(n+1,0)=0 \text{ for } x\in\{p,q,r,s\}$$

UNIVERSALITY $\longrightarrow$ ZERONESS

$$\boxed{L(\mathbb{A})=\mathbb{A}^* \text{ iff } \forall n,k.\ f_s(n,k)=S(n,k)}$$

$$f_p(n+1,k+1)=0$$

$$f_q(n+1,k+1)=f_p(n,k)+(k+1)\cdot f_p(n,k+1)+f_q(n,k)+k\cdot f_q(n,k+1)$$

$$f_r(n+1,k+1)=f_p(n,k)+(k+1)\cdot f_p(n,k+1)+f_r(n,k+1)$$

$$f_s(n+1,k+1)=f_q(n,k+1)+f_r(n,k)+k\cdot f_r(n,k+1)$$

FROM INCLUSION TO UNIVERSALITY

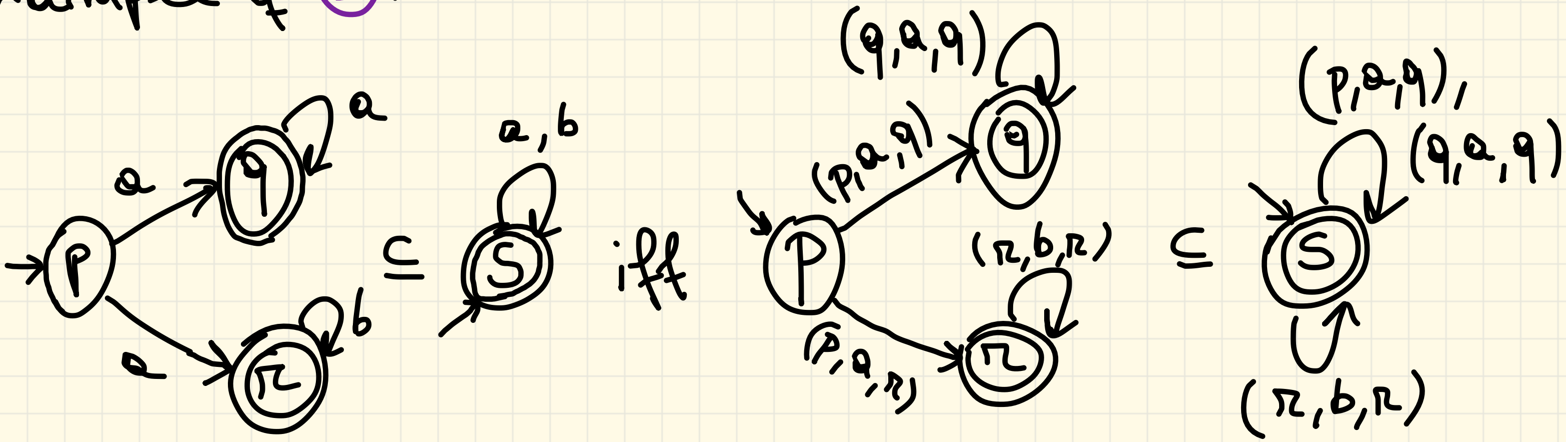
- Usual reduction from inclusion to emptiness: $L \subseteq M$ iff $L \cap M^c = \emptyset$.
- Inclusion "easy"/"hard" when M deterministic/non-deterministic. L irrelevant.
- Doesn't help if M cannot be efficiently complemented (UFA, UCFG, ...).
- Alternative reduction:

unamb. ① unamb. ② disjoint

$$\underset{\text{NFA}}{L} \subseteq \underset{\text{CFG}}{M} \text{ iff } L' \subseteq M' \text{ iff } \underbrace{\underbrace{M' \cap L'}_{\text{unamb.}} \cup \underbrace{(L')^c}_{\text{deterministic}}}_{\boxed{\text{unamb}}} = \Sigma^*$$

deterministic, poly-blowup

Example of ①:



REMARKS

- NFA ✓
- CFG ✓
- NRA ✓
- ⋮

LINREC SEQUENCES

shifts $(\sigma_1 f)(n,k) = f(n+1,k)$, $(\sigma_2 f)(n,k) = f(n,k+1)$.

EXAMPLE

$f_r(n+1,k+1) = f_p(n,k) + (k+1)\cdot f_p(n,k+1) + f_r(n,k+1)$ becomes

$\sigma_1\sigma_2 f_r = f_p + (k+1)\cdot\sigma_2 f_p + \sigma_2 f_r$

f is LINREC if $\exists\ f = f_1, f_2, \dots, f_m \in \mathbb{N}^2 \to \mathbb{Q}$:

$$\begin{cases} \sigma_1\sigma_2 f_1 = P_{11} f_1 + \dots + P_{1m} f_m \\ \quad\vdots \\ \sigma_1\sigma_2 f_m = P_{m1} f_1 + \dots + P_{mm} f_m \end{cases}$$

where $P_{ij} = p(n,k) + q(n,k)\sigma_1 + r(n,k)\sigma_2$, $p,q,r \in \mathbb{Q}[n,k]$

together with initial conditions $f_i(0,0), f_i(n+1,0), f_i(0,k+1) \in \mathbb{Q}$.

SEQUENCES ZOO

dim 1: $\mathbb{N} \to \mathbb{Q}$

growth $O((n!)^\beta)$

P-recursive

$(n+2)C(n+1) = (4n+2)C(n)$

Catalan numbers

ultimately per. modulo no prime [2]

Polyrec [1]

$f(n+1) = f(n)^2$

ultimately periodic mod. suff. large primes [1]

$O(2^{2^n})$

linrec

$$\begin{cases} f(n+1) = n \cdot g(n) + 3n^3 g(n) \\ g(n+1) = n^2 \cdot f(n) - g(n) \end{cases}$$

growth rate $O((n!)^\beta)$

C-recursive

$f(n+1) = f(n) + f(n-1)$

growth rate $O(\alpha^n)$

[1] Cadilhac, Mazowiecki, Paperman, Pilipczuk & Sénizergues, ICALP'20.
[2] Alter & Kubota, 1973.

SEQUENCES ZOO

<u>dim 2</u> : $\mathbb{N}\times\mathbb{N} \to \mathbb{Q}$

(P-recursive & linrec : already in dimension one.)

linrec ⊄ P-recursive:

- $n^k : \mathbb{N}^2 \to \mathbb{Q}$ is linrec (i.e., $f(n+1,k+1) = (n+1)\cdot f(n+1,k)$).
- its diagonal is n^n.
- $n^n : \mathbb{N} \to \mathbb{Q}$ is not P-recursive [1].
- P-recursive sequences closed under diagonal [2].

[1] : Flajolet, Gerhold, Salvy, "On the non-holonomic character...", 2005.

[2] : Lipshitz, "D-finite power series", 1989.

SKEW POLYNOMIALS [O. Ore 1930's]

COMMUTATION RULE

$$\sigma_1 \cdot p(n,k) = p(n+1,k)\cdot\sigma_1$$

COMMUTATION RULE

$$\sigma_2 \cdot p(n,k) = p(n,k+1)\cdot\sigma_2$$

Weyl algebras $W_1 = \mathbb{Q}[n,k][\sigma_1]^*$, $W_2 = \mathbb{Q}[n,k][\sigma_1][\sigma_2]$.

EXAMPLE

$$\begin{cases} f_p(n+1,k+1) = 0 \\ f_q(n+1,k+1) = \\ = f_p(n,k) + (k+1)\cdot f_p(n,k+1) + f_q(n,k) + k\cdot f_q(n,k+1) \\ f_r(n+1,k+1) = f_p(n,k) + (k+1)\cdot f_p(n,k+1) + f_r(n,k+1) \\ f_s(n+1,k+1) = f_q(n,k+1) + f_r(n,k) + k\cdot f_r(n,k+1) \\ S(n+1,k+1) = S(n,k) + (k+1)\cdot S(n,k+1) \\ g(n,k) = S(n,k) - f_s(n,k) \end{cases}$$

System of linear equations with W2 coefficients

$$\begin{aligned} &\sigma_1\sigma_2 f_p &= 0 \\ &-\sigma_2 f_p + (\sigma_1\sigma_2 - k\sigma_2 - 1) f_q &= 0 \\ &-\sigma_2 f_p + (\sigma_1\sigma_2 - \sigma_2) f_r &= 0 \\ &-\sigma_2 f_q - (1 + k\sigma_2) f_r + \sigma_1\sigma_2 f_s &= 0 \\ &(\sigma_1\sigma_2 - (k+1)\sigma_2 - 1) S &= 0 \\ &g - (S - f_s) &= 0 \end{aligned}$$

* Traditionally $W_1 = \mathbb{Q}[n,k][X;\sigma_1]$ with $X\cdot p = (\sigma_1 p)\cdot X$.

ZERONESS PROBLEM via ELIMINATION

$$\begin{aligned}
&\sigma_1\sigma_2 f_p && = 0 && eq_1\\
&-\sigma_2 f_p + (\sigma_1\sigma_2 - k\sigma_2 - 1) f_q && = 0 && eq_2\\
&-\sigma_2 f_p + (\sigma_1\sigma_2 - \sigma_2) f_r && = 0 && eq_3\\
&-\sigma_2 f_q - (1 + k\sigma_2) f_r + \sigma_1\sigma_2 f_s && = 0 && eq_4\\
&(\sigma_1\sigma_2 - (k+1)\sigma_2 - 1) S && = 0 && eq_5\\
&g - (S - f_s) && = 0 && eq_6
\end{aligned}$$

$\downarrow$ remove f_p & eq_1 by $eq_2' := \sigma_1 \cdot eq_2 + eq_1$,

$eq_3' := \sigma_1 \cdot eq_3 + eq_1$

$$\begin{aligned}
&(\sigma_1^2\sigma_2 - k\sigma_1\sigma_2 - 1) f_q && = 0 && eq2'\\
&+ (\sigma_1\sigma_2 - \sigma_2) f_r && = 0 && eq3'\\
&-\sigma_2 f_q - (1 + k\sigma_2) f_r + \sigma_1\sigma_2 f_s && = 0 && eq4\\
&(\sigma_1\sigma_2 - (k+1)\sigma_2 - 1) S && = 0 && eq5\\
&g - (S - f_s) && = 0 && eq6
\end{aligned}$$

ZERONESS PROBLEM via ELIMINATION

$$
\boxed{
\begin{aligned}
&(\sigma_1^2\sigma_2 - K\sigma_1\sigma_2 - 1) f_q &&= 0 && eq2' \\
&(\sigma_1\sigma_2 - \sigma_2) f_r &&= 0 && eq3' \\
&-\sigma_2 f_q - (1+K\sigma_2) f_r + \sigma_1\sigma_2 (S - g) &&= 0 && eq4' \\
&(\sigma_1\sigma_2 - (K+1)\sigma_2 - 1) S &&= 0 && eq5
\end{aligned}
}
$$

$\uparrow$ remove f_s & eq_6 by $eq4' = eq_4 - \sigma_1\sigma_2\, eq_6$

$$
\boxed{
\begin{aligned}
&(\sigma_1^2\sigma_2 - K\sigma_1\sigma_2 - 1) f_q &&= 0 && eq2' \\
&\qquad + (\sigma_1\sigma_2 - \sigma_2) f_r &&= 0 && eq3' \\
&\qquad -\sigma_2 f_q - (1+K\sigma_2) f_r + \sigma_1\sigma_2 f_s &&= 0 && eq4 \\
&(\sigma_1\sigma_2 - (K+1)\sigma_2 - 1) S &&= 0 && eq5 \\
&g - (S - f_s) &&= 0 && eq6
\end{aligned}
}
$$

ZERONESS PROBLEM via ELIMINATION

$$\boxed{\begin{aligned}
&(\sigma_1^2\sigma_2 - K\sigma_1\sigma_2 - 1) f_q &&= 0 \quad \text{eq } 2' \\
&(\sigma_1\sigma_2 - \sigma_2) f_\pi &&= 0 \quad \text{eq } 3' \\
&-\sigma_2 f_q - (1 + K\sigma_2) f_\pi + \sigma_1\sigma_2 (S - g) &&= 0 \quad \text{eq } 4' \\
&(\sigma_1\sigma_2 - (K+1)\sigma_2 - 1) S &&= 0 \quad \text{eq } 5
\end{aligned}}$$

remove f_π & eq_3' by finding $c, d \in W_2$: $\underbrace{c \cdot (\sigma_1\sigma_2 - \sigma_2) = d \cdot (1 + K\sigma_2)}_{\text{common left multiple}}$.

ZERONESS PROBLEM via ELIMINATION

$$\boxed{\begin{aligned}
&(\sigma_1^2\sigma_2 - K\sigma_1\sigma_2 - 1) f_q &&= 0 \quad \text{eq2}'\\
&(\sigma_1\sigma_2 - \sigma_2) f_r &&= 0 \quad \text{eq3}'\\
&-\sigma_2 f_q - (1+K\sigma_2) f_r + \sigma_1\sigma_2 (S-g) &&= 0 \quad \text{eq4}'\\
&(\sigma_1\sigma_2 - (K+1)\sigma_2 - 1) S &&= 0 \quad \text{eq5}
\end{aligned}}$$

$\downarrow$ remove f_r & eq_3' by finding $c, d \in W_2$: $\underbrace{c \cdot (\sigma_1\sigma_2 - \sigma_2) = d \cdot (1+K\sigma_2)}_{\text{common left multiple}}$.

$\text{eq}_4'' := c \cdot \text{eq}_3' - d \cdot \text{eq}_4'$.

$$\boxed{\begin{aligned}
&(\sigma_1^2\sigma_2 - K\sigma_1\sigma_2 - 1) f_q &&= 0 \quad \text{eq2}'\\
&-(\sigma_1^2\sigma_2^2 - \sigma_1\sigma_2^2) f_q + (\sigma_1^3 - \sigma_1^2)\cdot\sigma_2^2 (S-g) &&= 0 \quad \text{eq4}''\\
&(\sigma_1\sigma_2 - (K+1)\sigma_2 - 1) S &&= 0 \quad \text{eq5}
\end{aligned}}$$

can take

$c = 1 + (K+1)\sigma_2$,

$d = \sigma_1^2\sigma_2 - \sigma_1\sigma_2$.

ZERONESS PROBLEM via ELIMINATION

$$\begin{aligned}
&\Big[\big(-\sigma_1^5 + (2k+4)\sigma_1^4 - (k^2+5k+5)\sigma_1^3 + (k^2+3k+2)\sigma_1^2\big)\sigma_2^3 + \\
&\big(\sigma_1^4 - (k+2)\sigma_1^3 + (k+1)\sigma_1^2\big)\sigma_2^2\Big](S-g) &= 0 \qquad eq_4''' \\
&(\sigma_1\sigma_2 - (k+1)\sigma_2 - 1)S &= 0 \qquad eq_5
\end{aligned}$$

↑ remove f_q & eq_2' by finding $c, d \in W_2$: $c \cdot (-\sigma_1^2\sigma_2^2 + \sigma_1\sigma_2^2) = d \cdot (\sigma_1^2\sigma_2 - k\sigma_1\sigma_2 - 1)$.

let $eq_4''' := c \cdot eq_4'' - d \cdot eq_2'$.

can take $c = \sigma_1 - k - 1 - (\sigma_1 - k - 1) \cdot (\sigma_1 - k - 2)\sigma_2$,

$d = -(\sigma_1 - k - 1)(-\sigma_1 + 1)\sigma_2^2$.

$$\begin{aligned}
&(\sigma_1^2\sigma_2 - k\sigma_1\sigma_2 - 1) f_q &= 0 \qquad eq_2' \\
&-(\sigma_1^2\sigma_2^2 - \sigma_1\sigma_2^2) f_q + (\sigma_1^3 - \sigma_1^2)\cdot\sigma_2^2 (S-g) &= 0 \qquad eq_4'' \\
&(\sigma_1\sigma_2 - (k+1)\sigma_2 - 1)S &= 0 \qquad eq_5
\end{aligned}$$

ZERONESS PROBLEM via ELIMINATION

$$\Big[\big(-\sigma_1^5+(2k+4)\sigma_1^4-(k^2+5k+5)\sigma_1^3+(k^2+3k+2)\sigma_1^2\big)\sigma_2^3+\big(\sigma_1^4-(k+2)\sigma_1^3+(k+1)\sigma_1^2\big)\sigma_2^2\Big](S-g)=0 \qquad eq_4'''$$

$$(\sigma_1\sigma_2-(k+1)\sigma_2-1)S=0 \qquad eq_5$$

$\downarrow$ remove S & eq_5 by finding: $c\cdot \bullet = d\cdot \bullet$ *

$$\big(\sigma_1^5\sigma_2^4-(2k+8)\sigma_1^4\sigma_2^4-2\sigma_1^4\sigma_2^3+(k^2+9k+19)\sigma_1^3\sigma_2^4+(2k+8)\sigma_1^3\sigma_2^3+\sigma_1^3\sigma_2^2-(k^2+7k+12)\sigma_1^2\sigma_2^4-(2k+6)\sigma_1^2\sigma_2^3-\sigma_1^2\sigma_2^2\big)g=0$$

$\updownarrow$ the same as:

Cancelling relation

$$g(\underline{n+5},\underline{k+4})=(2k+8)g(n+4,k+4)+2g(n+4,k+3)-(k^2+9k+19)g(n+3,k+4)+$$
$$-(2k+8)g(n+3,k+3)-g(n+3,k+2)+(k^2+7k+12)g(n+2,k+4)+$$
$$+(2k+6)g(n+2,k+3)+g(n+2,k+2).$$

$-\ 5\times 4$

* Maple LDA package [Gerdt & Robertz '06].

ZERONESS PROBLEM via ELIMINATION

MAIN INGREDIENT : COMMON LEFT MULTIPLES

For every $a, b \in W_2 = \mathbb{Q}[n,k][\sigma_1][\sigma_2]$ there are $c, d \in W_2$: $c \cdot a = d \cdot b$.

→ follows from Euclidean pseudo-division for skew polynomial rings [1].

FIRST THEOREM

The zeroness problem for linrec sequences with UNIVARIATE polynomial coefficients in $\mathbb{Q}[k]$ is decidable.

→ Even with elementary complexity (better bounds will follow).

COROLLARY

Universality, equivalence & inclusion of unambiguous register automata without guessing are decidable.

[1]: O. Ore, "Linear equations in non-commutative fields", 1931.

ZERONESS PROBLEM via ELIMINATION

FIRST THEOREM

The zeroness problem for linrec sequences with UNIVARIATE polynomial coefficients in $\mathbb{Q}[k]$ is decidable.

INTUITION. Simple case:

pointwise maximal

1. $g(n+5,k+4)=(2k+8)g(n+4,k+4)+2g(n+4,k+3)-(k^2+9k+19)g(n+3,k+4)+$
$-(2k+8)g(n+3,k+3)-g(n+3,k+2)+(k^2+7k+12)g(n+2,k+4)+$
$+(2k+6)g(n+2,k+3)+g(n+2,k+2)$.

monic

g is zero iff g is zero on $\{0,\dots,5\}\times\{0,\dots,4\}$.

General case: only lex. maximal, not pointwise maximal

$(k^2-5)f_1(n+4,k+4)+(k-7)f_1(n+3,k+10)+(k^3+1)f_1(n+4,k+3)=0$

not monic

Ingredients: Lagrange's bound on roots, zeroness of f_1's sections.

ZERONESS IN EXPTIME

[1] Ore'31.
[2] Giesbrecht & Kim'13.

Ingredient 1: For univariate polynomial coefficients in $\mathbb{Q}[k]$,
commutative! $\mathbb{W}_2' = \mathbb{Q}[k,\sigma_1][\sigma_2]$ suffices (v.s. $\mathbb{W}_2 = \mathbb{Q}[n,k][\sigma_1][\sigma_2]$).

Ingredient 2: Construction of the rational skew field $\mathbb{Q}(k,\sigma_1)(\sigma_2)$ [1].

Ingredient 3: Hermite normal form of skew polynomial matrices $(\mathbb{W}_2')^{m\times m}$.

$$A = \begin{bmatrix} (\sigma_1 - 1)\sigma_2 & -\sigma_2 \\ -k\sigma_2 - 1 & \sigma_1\sigma_2 \end{bmatrix} \in (\mathbb{W}_2')^{2\times 2} \Rightarrow \underset{\text{HNF}}{H} = TA = \begin{bmatrix} 1 & \left(\frac{k}{\sigma_1 - 1} - \sigma_1\right)\sigma_2 \\ 0 & \sigma_2^2 - \frac{\sigma_2^1}{\sigma_1^2 - \sigma_1^1 - k - 1} \end{bmatrix}$$

→ TRIANGULAR

CANCELLING RELATION

$$\left(\sigma_2^2 - \frac{\sigma_2^1}{\sigma_1^2 - \sigma_1^1 - k - 1}\right) g = 0 \Rightarrow g(n+2, k+2) = g(n+1, k+2) + (k+1)\, g(n, k+2) + g(n, k+1).$$

Ingredient 4: HNF poly degrees & exp heights [2].

SECOND THEOREM

The zeroness problem for linrec sequences with UNIVARIATE polynomial coefficients in $\mathbb{Q}[k]$ is decidable in EXPTIME.

BACK TO UNAMBIGUOUS REGISTER AUTOMATA

THIRD THEOREM

The universality / equivalence / inclusion problems for unambiguous register automata are in 2-EXPTIME.

PROOF: Reduce to zeroness of linrec system of exp size.

COMMENTS:

- EXPTIME for fixed #registers.
- In $L(A) \subseteq L(B)$ it suffices for B to be unambiguous.

CONCLUSIONS

Cancelling relation: $\cancel{(k^2-5)}\,1\, f_1(n+4, k+4) + (k-7) f_1(n+3, k+10) = 0$

MONICITY CONJECTURE

There always exists a MONIC cancelling relation for linrec obtained from register automata.

- It holds in the handful of examples we tried.
- Consequences: zeroness in PTIME, univ./incl./eq. in EXPTIME.

FURTHER WORK

- Extend the counting approach to $(\mathbb{A}, <)$, classes of homogeneous structures, timed automata, pushdown register automata...
- Zeroness problem for other classes of sequences.

ZERONESS PROBLEM via ELIMINATION

FIRST THEOREM

The zeroness problem for linrec sequences with UNIVARIATE polynomial coefficients in $\mathbb{Q}[k]$ is decidable.

PROOF SKETCH. Eliminate all variables but f_1:

$$(k^2-5)f_1(n+4,k+4)+(k-7)f_1(n+3,k+10)+(k^3+1)f_1(n+4,k+3)=0$$

$$K := 1 + 4 + 2\cdot 5 = \text{bound on roots of lex. lead. coeff.} + \sigma_2\text{-order}.$$

1) $\forall (0\leq L\leq K)$. Section $f_1(n,L) : \mathbb{N}\to\mathbb{Q}$ is zero iff

$$f_1(0,L)=\dots=f_1(m\cdot(L+3),L)=0,\quad m=\text{order of } f_1(n,L).$$

Intuition: $f_1(n,L)$ is C-recursive of order $m\cdot(L+3)$.

2) $\forall (0\leq M\leq 4)$. section $f_1(M,k):\mathbb{N}\to\mathbb{Q}$ is zero iff

$$f_1(M,0)=\dots=f_1(M,d+2\cdot 5),\ d=\text{max } \sigma_2\text{-order of cancelling relation for } f_1(M,k).$$

3) f_1 is zero iff the sections in 1) & 2) are zero.