Formal verification - Tutorial 08

11/05/2026

Büchi's Büchi automata complementation

Let $D = \{\bot, 0, 1\}$ be a set with three elements. Intuitively, $\bot$ means no run, 0 means a run, and 1 means a run visiting an accepting transition. In the following, fix a set of states Q. A *graph* is a function $g \in D^{Q \times Q}$. A Büchi automaton is a tuple $A = (\Sigma, Q, I, \delta)$ where Σ is the input alphabet, Q is the set of states, $I \subseteq Q$ is the set of initial states, and the transition relation is represented by a family of graphs $\delta : \Sigma \to D^{Q \times Q}$, one graph δ_a for each input symbol $a \in \Sigma$. The language $L(A)$ recognised by A is defined as expected.

We study how to build the complement of A, following Büchi's original Ramsey-based construction. For two states $p, q \in Q$ and an input word $w \in \Sigma^*$, write

- $p \xrightarrow{w} q$ if there is a run from p to q on w,
- $p \xrightarrow{\;-w-\;} q$ if there is a run from p to q on w visiting an accepting transition, and
- $p \overset{w}{\not\to} q$ if there is no run from p to q on w.

An *edge* is a tuple $e = (p, x, q)$ where $p, q \in Q$ and $x \in D$. The *language* (or *semantics*) of an edge is defined as follows.

$$L(p, 1, q) = \left\{ w \in \Sigma^* \;\middle|\; p \xrightarrow{\;-w-\;} q \right\}, \tag{1}$$

$$L(p, 0, q) = \left\{ w \in \Sigma^* \;\middle|\; p \xrightarrow{w} q \right\} \setminus L(p, 1, q), \tag{2}$$

$$L(p, \bot, q) = \left\{ w \in \Sigma^* \;\middle|\; p \overset{w}{\not\to} q \right\} = \Sigma^* \setminus (L(p, 0, q) \cup L(p, 1, q)). \tag{3}$$

The language of a graph g is the intersection of the languages of its edges,

$$L(g) = \bigcap_{e \in g} L(e) = \bigcap_{p, q \in Q} L(p, g(p, q), q). \tag{4}$$

Finally, the language of two graphs g, h is

$$L(g, h) = L(g) \cdot L(h)^\omega. \tag{5}$$

We use languages $L(g, h)$ to construct the complement of $L(A)$.

Exercise 1. *Show that for every finite word $w \in \Sigma^*$ there exists a graph g_w, the* characteristic graph *associated to w, s.t. $w \in L(g_w)$.*

Solution. Let g_w be the graph defined as follows. For every pair of states $p, q \in Q$, we have $g_w(p,q) = 1$ if $p \overset{w}{\twoheadrightarrow} q$, $g_w(p,q) = 0$ if $p \xrightarrow{w} q$ but $p \overset{w}{\not\twoheadrightarrow} q$, and $g_w(p,q) = \perp$ if $p \overset{w}{\not\rightarrow} q$. By definition, w is in the language of every edge of g_w, and thus $w \in L(g_w)$, as required. □

Exercise 2. *Show that* $\Sigma^\omega = \bigcup_{g,h} L(g,h)$*, where the union is taken over all pairs of graphs* g, h*.*

Solution. We apply the infinite Ramsey's theorem. Consider an arbitrary infinite word $w = a_0 a_1 \cdots \in \Sigma^\omega$. Consider the infinite graph with nodes $\mathbb{N}$ and edges (i,j) for $i < j$ coloured by $g_{a_i a_{i+1} \cdots a_{j-1}}$. Since there are only finitely many colours, by the infinite Ramsey's theorem the graph as an infinite monochromatic subgraph. In other words, there exist an infinite set $I = \{i_0 < i_1 < \cdots\} \subseteq \mathbb{N}$ and a graph h s.t. for every i_j from I we have $g_{a_{i_j} a_{i_j+1} \cdots a_{i_{j+1}-1}} = h$. We can accordingly partition the word w into segments

$$w = w_0 w_1 w_2 \cdots$$

s.t. $g := g_{w_0}$ and $h = g_{w_1} = g_{w_2} = \cdots$. It follows that $w \in L(g,h)$, as required. □

Exercise 3. *For every pair of graphs* g, h*, show that either* $L(g,h) \subseteq L(A)$ *or* $L(g,h) \cap L(A) = \varnothing$*.*

Solution. Intuitively, all words in $L(g)$ share the same behaviour in A. We show that $L(g,h) \cap L(A) \neq \varnothing$ implies $L(g,h) \subseteq L(A)$. To this end, consider an arbitrary word $w' \in L(g,h) = L(g) \cdot L(h)^\omega$ which is naturally factorised as $w' = w'_0 w'_1 w'_2 \cdots$ where $w'_0 \in L(g)$ and $w'_1, w'_2, \cdots \in L(h)$. By assumption, there exists a word $w \in L(g,h) = L(g) \cdot L(h)^\omega$ which is accepted by the automaton A. Thus let π be an accepting run of A on w. We can write $w = w_0 w_1 w_2 \cdots$ where $w_0 \in L(g)$ and $w_1, w_2, \cdots \in L(h)$. We can split π into segments

$$\pi = p_0 \xrightarrow{w_0} p_1 \xrightarrow{w_1} p_2 \xrightarrow{w_2} \cdots$$

s.t. infinitely many segments visit an accepting transition. Since w_0, w'_0 are both in $L(g)$, they share the same behaviour in A, and in particular $p_0 \xrightarrow{w'_0} p_1$. The same argument applies to $w_i, w'_i \in L(h)$, which means that $p_i \xrightarrow{w'_i} p_{i+1}$ as well. We thus obtain an infinite run

$$\pi = p_0 \xrightarrow{w'_0} p_1 \xrightarrow{w'_1} p_2 \xrightarrow{w'_2} \cdots$$

Moreover, the construction preserves accepting segments, and thus π is an accepting run of A on w', as required. □

Exercise 4. *Show that the following problem is decidable, in polynomial time. In input we are given a pair of graphs* g, h *and we need to decide whether* $L(g,h) \cap L(A) \neq \varnothing$*.*

Solution. Decompose h into its strongly connected components (SCCs). Call a component *accepting* if it contains an accepting transition. This can be done in polynomial time. We have $L(g,h) \cap L(A) \neq \varnothing$ iff there exists an edge $e = (p, x, q)$ of g with $p \in I$ and $x \neq \perp$ s.t. in h there is a path from q to an accepting SCC of h. This can also be checked in polynomial time. □

Exercise 5. *For every pair of graphs g, h, the language $L(g, h)$ is ω-regular.*

Solution. First of all, the language of a single edge $L(e) \subseteq \Sigma^*$ is regular, as it easily follows from its definition. The language of a graph is defined as a finite intersection of languages of edges, and is therefore regular. Finally, $L(g, h)$ is defined as a concatenation of a regular language $L(g)$ and an ω-power of a regular language $L(h)$, and is therefore ω-regular. □

The exercises above show that we can write $\Sigma^\omega \setminus L(A)$ *effectively* as a finite union of ω-regular languages $L(g, h)$:

$$\Sigma^\omega \setminus L(A) = \bigcup_{g,h : L(g,h) \cap L(A) = \varnothing} L(g, h). \tag{6}$$

References