Formal verification - Tutorial 09

18/05/2026

Gale-Stewart games

Exercise 1. *Consider Gale-Stewart games with context-free winning conditions.*

- *Can we decide whether a given player has a winning strategy?*

Solution. No, GS games with ω-context-free winning conditions are undecidale. We can reduce from the universality problem for ω-context-free languages, which is known to be undecidable. For instance, consider an ω-CFL $L \subseteq \Sigma^\omega$. Consider the GS game where P1 plays letters from Σ and P2 plays letters from a singleton alphabet $\{a\}$. Consider the winning condition for the second player $W_{II} := \{a_0\$a_1\$a_2\$\cdots \mid a_0a_1a_2\cdots \in L\}$. By the closure properties of ω-CFLs, W_{II} is an ω-CFL. Then P2 has a winning strategy iff $L = \Sigma^\omega$.

A similar argument can be used to show undecidability when considering a ω-CFL winning condition W_I for P1. □

Closure under uniformisation

Exercise 2 ([1, Sec. 6])**.** *Consider Presburger arithmetic* $(\mathbb{N}, +, \leqslant)$*. Construct a formula* $\varphi(X, Y)$ *with two monadic free variables* X, Y *s.t.* $\exists Y \forall X \varphi(X, Y)$ *is true, but there is no* recursive *witness* $Y = \tilde{Y}$ *s.t.* $\forall X \varphi(X, \tilde{Y})$ *is true.*

Solution. Let $\varphi_{\text{sq}}(X)$ say that X is the set of squares $S := \{0^2, 1^2, 2^2, 3^2, \dots\}$. It is definable in Presburger arithmetic since the distance of three subsequent squares $a = n^2$, $b = (n+1)^2$, $c = (n+2)^2$ increases by two: $(c-b)-(b-a) = 2$. We can then write $\varphi_{\text{sq}}(X)$ as the conjunction of the following three formulas:

- $X(0) \wedge X(1)$,
- X is infinite, and
- "the distance of three consecutive elements of X increases by two" (as described above).

Since multiplication is definable in the structure $(\mathbb{N}, +, S)$, there exits a non-recursive set $M \subseteq \mathbb{N}$ definable by a formula $\psi(S, y)$ over $(\mathbb{N}, +, S)$ with one free variable y. Now consider the following Presburger formula

$$\varphi(X, Y) \equiv \varphi_{\text{sq}}(X) \rightarrow \forall y.(Y(y) \leftrightarrow \psi(X, y)).$$

Since there is a unique set of squares, the only solution to the formula above is $Y = M$, which is not recursive. □

Deterministic separability

Consider language $L, M, S \subseteq \Sigma^\omega$. We say that S *separates* L and M if $L \subseteq S$ and $M \cap S = \emptyset$.

Exercise 3. *Show that the following problem is decidable. In input we are given two nondeterministic Büchi automata A, B over the same alphabet Σ and a finite set of priorities $P \subseteq \mathbb{N}$, and we need to decide whether there is a deterministic parity automaton S over Σ with priorities in P s.t. $L(S)$ separates $L(A)$ and $L(B)$.*

References

[1] Wolfgang Thomas. *Facets of Synthesis: Revisiting Church's Problem*, page 1–14. Springer Berlin Heidelberg, 2009. URL: http://dx.doi.org/10.1007/978-3-642-00596-1_1, doi:10.1007/978-3-642-00596-1_1.