# Formal verification - Tutorial 10

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## Rational series

Let $\Sigma$ be a finite alphabet. A *series* is a mapping $f : \Sigma^* \to \mathbb{Q}$. The *support* of a series $f$ is the language $\mathsf{supp}(f) = \{w \in \Sigma^* \mid f(w) \neq 0\}$. Recall that the set of rational series coincides with the set of series recognised by weighted finite automata.

**Exercise 1.** *Show an example of a rational series with a non-regular support.*

*Solution.* Consider the series mapping a word $w \in \Sigma^*$ to the difference of the number of occurrences of $a$ and $b$ in $w$. The support of this series is the language of words with an unequal number of $a$'s and $b$'s, which is well-known not to be regular. The series is rational, in fact a weighted automaton of dimension two suffices to recognise it. □

The *left shift* (or *left quotient*, or *left derivative*) of a series $f$ by a word $w \in \Sigma^*$ is the series $w^{-1}f$ defined by $(w^{-1}f)(u) = f(wu)$ for all $u \in \Sigma^*$. The *rank* of a series $f$ is the dimension of the vector space generated by the set of all left shifts of $f$, i.e., the set $\{w^{-1}f \mid w \in \Sigma^*\}$. Recall that the rank of a series is finite if, and only if, the series is rational.

**Exercise 2.** *Assume that $f, g$ are rational series with rank $k$, resp., $\ell$.*

1. *Show that the rank of $f + g$ is at most $k + \ell$.*
2. *The* Hadamard product *of $f$ and $g$ is the series $f \odot g$ defined by $(f \odot g)(w) = f(w)g(w)$ for all $w \in \Sigma^*$. Show that the rank of $f \odot g$ is at most $k \cdot \ell$.*

A weighted automaton is *observable* at an initial condition $u \in \mathbb{Q}^{1\times k}$ if, for every initial condition $u' \in \mathbb{Q}^{1\times k}$, $[\![A_u]\!] = [\![A_{u'}]\!]$ implies $u = u'$. A weighted automaton is *observable* if it is observable at every initial condition.

**Exercise 3.** *Show that the observability problem for weighted automata is decidable in polynomial time.*

A *parametric weighted automaton* is a tuple

$$A = (\Sigma, k, \ell, u, M, v)$$

where

- $\Sigma$ is a finite input alphabet,

- $k \in \mathbb{N}$ is the dimension of the state
- $\ell \in \mathbb{N}$ is the number of parameters,
- $u \in \mathbb{Q}[p_1, \ldots, p_\ell]^{1\times k}$ is the parametric initial vector,
- $M : \Sigma \to \mathbb{Q}[p_1, \ldots, p_\ell]^{k\times k}$ is the *transition function* mapping each letter to a parametric transition matrix, and
- $v \in \mathbb{Q}[p_1, \ldots, p_\ell]^{k\times 1}$ is the parametric final vector.

For every given valuation for the parameters $\nu \in \mathbb{Q}^\ell$ we obtain a concrete weighted automaton $A(\nu)$ in the expected way. This concrete automaton will then recognise the rational series $[\![A(\nu)]\!] : \Sigma^* \to \mathbb{Q}$.

**Exercise 4.** *Show that the set of all parameter valuations $\nu \in \mathbb{C}^\ell$ s.t. $A(\nu)$ recognises the zero series is an algebraic variety. Show that polynomial generators of degree polynomial in $k$ suffice to define this variety.*

The *parameter identifiability problem* for a parametric weighted automaton $A$ amounts to deciding whether, for all parameter valuations $\nu, \nu' \in \mathbb{C}^\ell$, $[\![A(\nu)]\!] = [\![A(\nu')]\!]$ implies $\nu = \nu'$.

**Exercise 5.** *Show that the parameter identifiability problem is decidable for parametric weighted automata.*