Formal verification - Tutorial 11

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Markov chains

Exercise 1. *Let $M = (S, \rightarrow)$ be a finite Markov chain and consider a target set $T \subseteq S$. For every initial state $s \in S$, let*

$$x_s = \mathbb{P}(M, s \models \mathbf{F}\, T)$$

be the probability of reaching a state in T when started from s.

1. *Can we compute the set of initial states s such that $x_s = 0$ without solving linear equations?*
2. *Can we compute the set of initial states s such that $x_s = 1$ without solving linear equations?*

Does the answers change if M is infinite-state?

Solution. Regarding the first question, we note that $x_s = 0$ if, and only if, there is no path from s to any state in T. Since paths are finite, this observation holds even for infinite-state Markov chains. It follows that we can compute the set of states s such that $x_s = 0$ by performing a graph search on the underlying graph of M.

Regarding the second question, we note that the set of states that appear infinitely often in a run of M is an end component of M. We can thus decompose M into its end components $C_1, \ldots, C_n$ (which are the bottom strongly connected components of the underlying graph). We then have that $x_s = 1$ if, and only if, every end component reachable from s intersects T.

The answer for the second question changes for infinite-state Markov chains. For instance consider the one-sided random walk with reachability probability satisfying $x = 1 - p + p \cdot x^2$. Whether $x = 1$ depends on the actual value of p, not just on whether p is zero or not. □

Exercise 2. *Let $M = (S, \rightarrow)$ be a finite Markov chain and consider two sets $I, T \subseteq S$. For every initial state $s \in S$, let*

$$x_s = \mathbb{P}(M, s \models I\, \mathbf{U}\, T)$$

be the probability of reaching a state in T while staying in I when started from s.

1. *Provide a characterisation of the vector of probabilities $(x_s)_{s \in S}$.*
2. *Can we compute these probabilities?*

Solution. We reduce the problem to that of computing reachability probabilities when $I = S$ (trivial invariant). First, make all states in T and in $S \backslash (I \cup T)$ absorbing. Clearly, this operation does not change the probabilities x_s. In the new chain, compute the probability of reaching T. This will be the same as the probability of reaching T while staying in I in the original chain. The reason is that in the original chain we can never step out of I while trying to reach T. In the new chain, we model this by making T unreachable when stepping out of I. □

Exercise 3. *Let M be a Markov chain and let L be an ω-regular property of M recognised by a nondeterministic Büchi automaton A. Can we decide efficiently*

1. *whether $\mathbb{P}(M \models L) = 0$?*
2. *whether $\mathbb{P}(M \models L) = 1$?*

Solution. First of all, notice that both queries are decidable, in PSPACE, by converting A into a deterministic parity automaton B (with a singly exponential blow-up) and then computing the probability that M satisfies the parity condition of D (in NC^2).

We show that $\mathbb{P}(M \models L) = 1$ is PSPACE-hard. We reduce from the universality problem for nondeterministic finite automata, which is well-known to be PSPACE-hard. Let B be a nondeterministic finite automaton over alphabet $\Sigma = \{a, b\}$. We construct a Markov chain M with three states $a, b, \$$ and transitions

$$
\begin{aligned}
& a \xrightarrow{1/3} a, a \xrightarrow{1/3} b, a \xrightarrow{1/3} \$, \\
& b \xrightarrow{1/3} a, b \xrightarrow{1/3} b, b \xrightarrow{1/3} \$, \\
& \$ \xrightarrow{1} \$.
\end{aligned}
$$

We construct a nondeterministic Büchi automaton A recognising $L(A) = L(B) \cdot \$^\omega$. We claim that $\mathbb{P}(M \models L) = 1$ if, and only if, B is universal ($L(B) = \Sigma^*$). The "only if" direction is easy: If B is universal, then $\mathbb{P}(M \models L) = 1$ since all runs of M will eventually almost surely reach \$ and thus satisfy L. For the "if" direction, assume that B is not universal, and let $w \in \Sigma^* \backslash L(B)$. We have that the whole cylinder set $w\Sigma^\omega$ is disjoint from $L(A)$. This set has measure $1/3^{|w|}$, since M generates words uniformly at random, and thus $\mathbb{P}(M \models L) < 1 - 1/3^{|w|} < 1$, as required. □

Exercise 4. *Let M be a Markov chain and let L be an ω-regular property of M recognised by an unambiguous Büchi automaton A. Show that $\mathbb{P}(M \models L)$ can be computed in PTIME.*

Solution. We pretend that A is deterministic and we compute the product $M \times A$ of M and A. The product contains a weighted edge

$$
(s, p) \xrightarrow{x} (t, q)
$$

when $s \xrightarrow{x} t$ is a transition of M (with probability x) and $p \xrightarrow{s} q$ is a transition of A reading input symbol s. Since A is nondeterministic, the weight of edges outgoing from (s, p) may sum up to more than 1. Never mind that. Decompose

$M \times A$ into its bottom strongly connected components and let C be the union of all the bottom strongly connected components that contain an accepting state of A. Consider the least nonnegative solution to the system of equations

$$\begin{aligned} x_{s,p} &= 1 && \text{if } (s,p) \in C, \text{ and otherwise} \\ x_{s,p} &= \sum_{(s,p) \xrightarrow{x} (t,q)} x \cdot x_{t,q}. \end{aligned}$$

Since A is unambiguous (accepted words have a unique accepting run), $x_{s,p} \in [0,1]$, and in $\mathbb{Q}$. Then, $\mathbb{P}(M \models L) = \sum_{s \in S} x_{s,p_0}$, where p_0 is the initial state of A. ☐

Random walks

Exercise 5. *Let* $p \in [0,1]$ *be a parameter. Consider the random walk induced by the 1-state pPDS over stack alphabet* $\Gamma = \{X\}$ *and transitions*

$$X \xrightarrow{1-p} \varepsilon \quad \textit{and} \quad X \xrightarrow{p} X^3.$$

Compute the probability of termination from configuration X *as a function of* p*. For which values of* p *is this probability equal to* 1*?*

Solution. The required probability x is the least nonnegative solution of

$$x = 1 - p + p \cdot x^3.$$

We can solve this equation and obtain that x is the minimum of 1 and (for $p > 0$)

$$\frac{-p + \sqrt{4p - 3p^2}}{2p}$$

This is a monotonically decreasing function of p that is equal to 1 when $p = 1/3$. ☐

Exercise 6. *Let* $n \in \mathbb{N}$ *and* $p \in [0,1]$ *be parameters. Consider the random walk induced by the 1-state pPDS over stack alphabet* $\Gamma = \{X\}$ *and transitions*

$$X \xrightarrow{1-p} \varepsilon \quad \textit{and} \quad X \xrightarrow{p} X^n.$$

Show that the probability of termination x *from configuration* $X\varepsilon$ *is* 1 *if, and only if,* $p \leqslant 1/n$*.*

Solution. The expected drift of the random walk is

$$(1 - p) \cdot (-1) + p \cdot (n - 1) = p \cdot n - 1.$$

If $p < 1/n$, then the expected drift is negative, and thus $x = 1$. If $p > 1/n$, then the expected drift is positive, and thus $x < 1$. If $p = 1/n$ (critical value), then the expected drift is zero, and we need to argue differently. The required probability x is the least nonnegative solution of

$$x = 1 - p + p \cdot x^n.$$

In other words we have $x^n - \frac{1}{p} \cdot x + \frac{1-p}{p} = 0$. The value $x = 1$ is always a solution of this equation, regardless of the value of p. We thus divide the polynomial by $x - 1$ and obtain

$$1 - p \cdot (1 + x + \cdots + x^{n-1}) = 0.$$

This equation has solution $x = 1$ if, and only if, $p = 1/n$. □

Probabilistic pushdown automata

Exercise 7. *Show that the quantitative reachability problem for pPDA is decidable with PSPACE complexity.* Hint: Use the fact that the existential fragment of Tarki's arithmetic is decidable in PSPACE.

Solution. We can massage the logical sentence to have only universal quantification, and then we can negate it to obtain an existential sentence. □

Exercise 8. *Can we decide efficiently whether* $[pXq] = 0$*? And whether* $[pXq] = 1$*?*

Solution. The first query $[pXq] = 0$ is decidable in PTIME. Its complement, $[pXq] > 0$, is equivalent to reachability $pX \to^* q\varepsilon$ in the underlying pushdown system, which is decidable in PTIME. □

References